TRINITY COLLEGE

Semester One Examination, 2021

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

SN108-175-4

Section Two: Calculator-assumed **65% (98 Marks)**

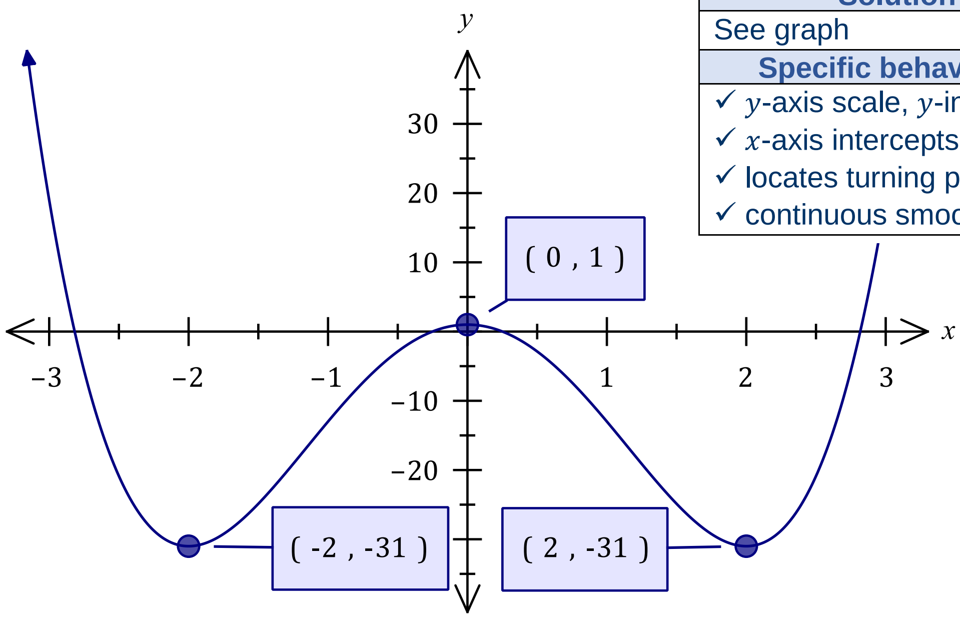
This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(8 marks)**

Let $f(x) = 2x^4 + ax^2 + 1$.

(a) Sketch the graph of $y = f(x)$ when $a = -16$. (4 marks)



Solution
See graph
Specific behaviours
✓ y-axis scale, y-intercept
✓ x-axis intercepts
✓ locates turning points
✓ continuous smooth curve

(b) Show that the graph of $y = f(x)$ will always have a maximum turning point at $x = 0$ if $a < 0$. (4 marks)

Solution
$f'(x) = 8x^3 + 2ax$ $f'(0) = 0$ Hence curve always stationary when $x = 0$. $f''(x) = 24x^2 + 2a$ $f''(0) = 2a$ If $a < 0$ then $f''(0) < 0$ and so the curve will always be concave down. Hence a maximum at $x = 0$.
Specific behaviours
✓ shows $f'(0) = 0$ ✓ states always stationary when $x = 0$ ✓ shows $f''(0) = 2a$ ✓ justifies maximum using second derivative

Question 10 **(6 marks)**

(a) List A contains the digits in the first 200 decimal places of π. The relative frequencies of the digits are:

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	0.095	0.100	0.120	0.095	0.110	0.100	0.080	0.060	0.125	0.115

Determine the probability that a randomly selected digit from list A

(i) is odd. (1 mark)

Solution

$$P = 0.1 + 0.095 + 0.1 + 0.06 + 0.115$$
$$= 0.47$$

Specific behaviours

✓ correct probability

(ii) is a factor of 18, given that it is not odd. (2 marks)

Solution

$$P = \frac{0.12 + 0.08}{1 - 0.47} = \frac{20}{53} \approx 0.3774$$

Specific behaviours

✓ numerator
✓ denominator and simplifies

(b) The discrete random variable X is defined by

$$P(X = x) = \begin{cases} {}^{1}/_{6} & x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the expected value and exact variance of X. (3 marks)

Solution

$$E(X) = \frac{15}{6} = \frac{5}{2} = 2.5$$
$$SD(X) = 8.51$$
$$\text{Var}(X) = 2.92$$

Specific behaviours

✓ $E(X)$
✓ indicates calculation or use of CAS
✓ exact variance

Question 11 **(7 marks)**

A hot potato was removed from an oven and placed on a cooling rack. Its temperature T, in degrees Celsius, t minutes after being removed from the oven was modelled by

$$T = 23 + 187e^{kt}.$$

The temperature of the potato at $t = 8$ is half the initial temperature.

(a) Determine the value of the constant k. (3 marks)

Solution
$T_0 = 23 + 187 = 210$ $105 = 23 + 187e^{8k} \Rightarrow k = -0.103$
Specific behaviours
✓ indicates initial temperature ✓ equation for temperature halving ✓ solves for k

(b) The temperature of the potato eventually reached a steady state. Determine the time taken for its temperature to first fall to within $1\ °C$ of this steady state. (2 marks)

Solution
$T_\infty = 23$ $24 = 23 + 187e^{-0.103t} \Rightarrow t = 50.8 \text{ minutes}$
Specific behaviours
✓ indicates steady state temperature ✓ correct time, to at least 1 dp

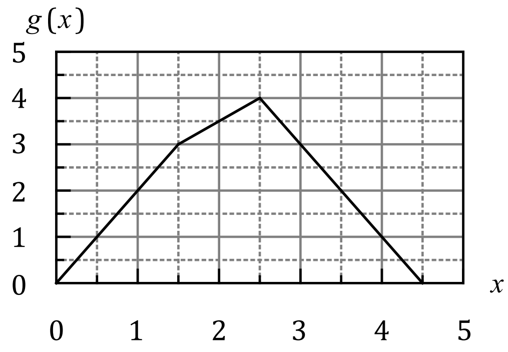
(c) Determine the time at which the potato was cooling at a rate of $1\ °C$ per minute. (2 marks)

Solution
$\frac{dT}{dt} = 19.261e^{-0.103t}$ $19.261e^{-0.103t} = -1 \Rightarrow t = 28.7 \text{ minutes}$
Specific behaviours
✓ indicates derivative ✓ correct time, to at least 1 dp

Question 12 **(8 marks)**

The graph of function g, and a table of values for function f and its derivatives are shown below.

x	1	2	3
$f(x)$	2	1	3
$f'(x)$	3	2	2
$f''(x)$	-1	-2	1



(a) Evaluate $h'(k)$ when

(i) $h(x) = g\big(f(x)\big)$ and $k = 3$. (3 marks)

Solution
$h'(3) = g'\big(f(3)\big) \times f'(3)$ $= g'(3) \times 2$ $= -2 \times 2 = -4$
Specific behaviours
✓ correct application of chain rule   ✓ correct value for $g'(x)$ ✓ correct value

(ii) $h(x) = f(x) \div$ (3 marks)

Solution
$h'(1) = \dfrac{f'(1)g(1) - f(1)g'(1)}{g(1)^2}$ $= \dfrac{(3)(2) - (2)(2)}{(2)^2} = \dfrac{1}{2}$
Specific behaviours
✓ correct application of quotient rule   ✓ correct values for $g(x)$ and $g'(x)$ ✓ correct value

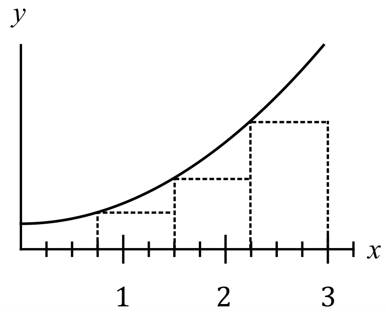
(b) Evaluate $h''(2)$ when $h'(x) = f'(x) \times g'(x)$. (2 marks)

Solution
$h''(2) = f''(2)g'(2) + f'(2)g''(2)$ $= (-2)(1) + (2)(0)$ $= -2$
Specific behaviours
✓ uses product rule with at least two correct values   ✓ correct result

Question 13 **(8 marks)**

The graph of $y = f(x)$ is shown at right with 3 equal width inscribed rectangles. An estimate for the area under the curve between $x = 0.75$ and $x = 3$ is required.

The function f is defined as $f(x) = 4x^2 + 5$ and let the area sum of the 3 rectangles be S_3.



(a) State the height and width of the second rectangle. (1 mark)

Solution
height = 14   width = 0.75
Specific behaviours
✓ correctly states all values

(b) Calculate the value of S_3. (3 marks)

Solution
$S_3 = 0.75\big((4(0.75)^2 + 5) + (4(1.5)^2 + 5) + (4(2.25)^2 + 5)\big)$ $= 0.75(7.25 + 14 + 25.25)$ $= 0.75(46.5)$ $= \frac{279}{8} = 34.875\ \text{u}^2$
Specific behaviours
✓ indicates correct calculation for one rectangle ✓ correct heights of all rectangles ✓ correct value

(c) Explain, with reasons, how the value of δx and the area estimate S_n will change as the number of inscribed rectangles increase. marks)

Solution
dx is the width of each rectangle and so must decrease. S_n will increase, approaching true area under curve, as area 'lost' between curve and rectangles will decrease.
Specific behaviours
✓ indicates δx will decrease as it's the rectangle width ✓ indicates S_n will increase

(d) Determine the limiting value of S_n as $n \to \infty$. (2 marks)

Solution
$S_\infty = \int_{0.75}^{3} f(x)\,dx = \frac{747}{16} = 46.6875\ \text{u}^2$
Specific behaviours
✓ correct integral ✓ correct limiting value

Question 14 **(8 marks)**

The area A of a regular polygon with n sides of length x is given by

$$A = \frac{n\,x^2 \cos\left(\frac{\pi}{n}\right)}{4\sin\left(\frac{\pi}{n}\right)}$$

(a) Simplify the above formula when $n = 6$ to obtain a function for the area of a regular hexagon. (2 marks)

Solution
$A = \frac{6\,x^2 \cos\left(\frac{\pi}{6}\right)}{4\sin\left(\frac{\pi}{6}\right)} = \frac{3\sqrt{3}x^2}{2}$
Specific behaviours
✓ correctly substitutes ✓ simplified function

(b) Use the increments formula to estimate the change in area of a regular hexagon when its side length increases from 10 cm to 10.5 cm. (3 marks)

Solution
$\frac{dA}{dx} = 3\sqrt{3}x, \quad x = 10, \quad \delta x = 0.5$ $\delta A \approx \frac{dA}{dx}\delta x$ $\approx 3\sqrt{3}(10)(0.5)$ $\approx 15\sqrt{3} \approx 25.98\text{ cm}^2$
Specific behaviours
✓ derivative of A with respect to x ✓ correct use of increments formula ✓ calculates change

(c) Use the increments formula to estimate the change in area of a regular polygon with sides of length 10 cm when its number of sides increases from 29 to 31. (3 marks)

Solution
$\frac{dA}{dn}\Big\vert_{n=29} \approx 461.55 \quad \delta n = 2 \quad x = 10$ $\delta A \approx \frac{dA}{dn}\delta n \approx 461.55 \times 2 \approx 923\text{ cm}^2$
Specific behaviours
✓ derivative of A with respect to n (CAS) ✓ indicates use of correct values of $n, \delta n, x$ ✓ calculates change

Question 15 **(8 marks)**

(a) It is known that 17% of a large number of smoke alarms in a complex of buildings are faulty. If an electrician randomly selects 8 alarms for inspection, determine

(i) the probability that none of the alarms will be faulty. (2 marks)

Solution
Let X be the number of faulty alarms. Then $X \sim B(8, 0.17)$. $P(X = 0) = 0.2252$
Specific behaviours
✓ defines distribution ✓ states probability

(ii) the probability that more than three alarms are faulty, given that at least one is faulty. (2 marks)

Solution
$P(X \geq 4) = 0.0328$ $P(X \geq 4 \vert P(X \geq 1) = \dfrac{0.0328}{1 - 0.2252} = 0.0423$
Specific behaviours
✓ indicates $P(X \geq 4)$ ✓ calculates conditional probability

(iii) the standard deviation of the distribution of the number of faulty alarms. (1 mark)

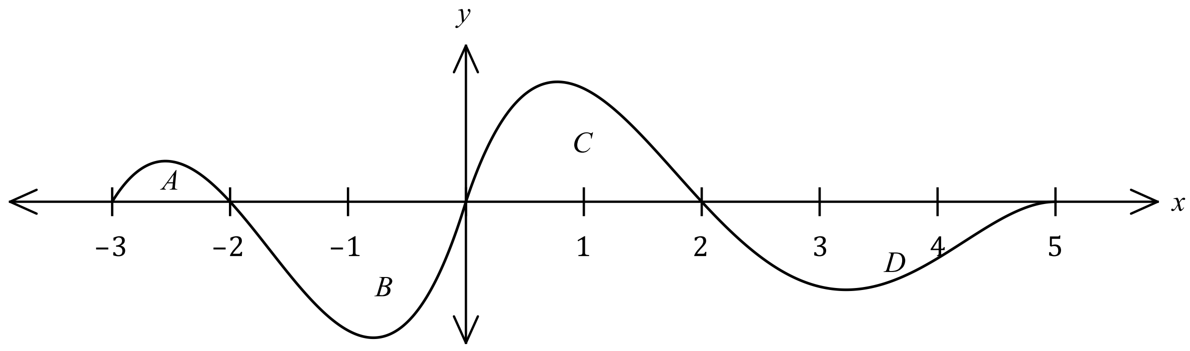
Solution
$sd = \sqrt{8 \times 0.17 \times 0.83} = 1.062$
Specific behaviours
✓ correct value

(b) In a newer complex that also has a large number of smoke alarms, only 7% are faulty. Determine, with reasoning, the minimum number of alarms that should be inspected so that the probability that at least one of them will be faulty is more than 99%. (3 marks)

Solution
$Y \sim B(n, 0.07)$ $P(Y \geq 1) \geq 0.99 \Rightarrow P(Y = 0) < 0.01$ $P(Y = 0) = (0.93)^n$ $0.93^n < 0.01$ $n = 64$
Specific behaviours
✓ identifies distribution and required probability ✓ expression for no faulty alarms, in terms of n ✓ correct number

Question 16 **(7 marks)**

Regions A, B, C and D bounded by the curve $y = f(x)$ and the x-axis are shown on this graph:



The areas of A, B, C and D are $9, 30, 23$ and 21 square units respectively.

(a) Determine the value of

(i) $\int_0^5 f(x)\,dx.$ (1 mark)

Solution
$I = 23 - 21 = 2$
Specific behaviours
✓ correct value

(ii) $\int_{-3}^{2} 5f(x)\,dx.$ (2 marks)

Solution
$I = 5(9 - 30 + 23) = 5(2) = 10$
Specific behaviours
✓ shows sum of signed areas
✓ uses linearity to obtain correct value

(iii) $\int_{-2}^{5} (f(x) - 4)\,dx.$ (2 marks)

Solution
$I = [-30 + 23 - 21] - [4(7)]$ $I = -28 - 28 = -56$
Specific behaviours
✓ uses linearity to obtain two integrals
✓ correct value

(b) Explain why $\int_{-2}^{2} f'(x)\,dx = 0.$ (2 marks)

Solution
Using fundamental theorem, result is $f(2) - f(-2)$. Since $f(-2) = f(2) = 0$, then the difference is 0.
Specific behaviours
✓ uses fundamental theorem to obtain result
✓ explains value of 0 using the two roots

Question 17 **(8 marks)**

The volume, V litres, of fuel in a tank is reduced between $t = 0$ and $t = 48$ minutes so that

$$\frac{dV}{dt} = -175\pi \sin\left(\frac{\pi t}{48}\right)$$

(a) Determine, to the nearest litre, the amount of fuel emptied from the tank

(i) in the first minute. (3 marks)

Solution
$\Delta V = \int_0^1 V'\,dt$ $= -17.985$ Hence 18 litres were emptied.
Specific behaviours
✓ writes integral for change   ✓ evaluates integral   ✓ answers as positive number of litres

(ii) in the last 7 minutes. (1 mark)

Solution
$\Delta V = \int_{41}^{48} V'\,dt = -866.3$ Hence 866 litres were emptied.
Specific behaviours
✓ correct number of litres

The tank initially held 18 600 litres of fuel.

(b) Determine the volume of fuel in the tank 5 minutes after the volume in the tank reached 12 000 litres. (4 marks)

Solution
$\int_0^T V'\,dt = -6\,600$ $T = 20.70$ $\Delta V = \int_{20.7}^{25.7} V'\,dt$ $= -2\,733$ $V(25.7) = 12\,000 - 2\,733$ $= 9\,267\text{ L}$
Specific behaviours
✓ equation for $\Delta V = -6\,600$ ✓ determines T ✓ determines ΔV ✓ correct volume

Alternative Solution
$V(t) = \int V'\,dt = 8400\cos\left(\frac{\pi t}{48}\right) + c$ $V(0) = 18\,600 \Rightarrow c = 10\,200$ $V(T) = 12\,000 \Rightarrow T = 20.70$ $V(25.7) = 9\,267\text{ L}$
Specific behaviours
✓ antiderivative for $V(t)$ ✓ determines c ✓ determines T ✓ correct volume

Question 18 **(8 marks)**

When an electronic device is run, it randomly generates one of the first four triangle numbers. The discrete random variable X is the number generated in one run of the device and the table below shows its probability distribution.

x	1	3	6	10
$P(X=x)$	a	b	0.2	0.3

E(X) = 5.6.

(a) Determine the value of the constant a and the value of the constant b. (3 marks)

Solution
Sum of probabilities: $$a+b+0.5=1$$ Mean: $$a+3b+4.2=5.6$$ Solving simultaneously: $$a=0.05, \qquad b=0.45$$
Specific behaviours
✓ equation using sum ✓ equation using mean ✓ both correct values

(b) The electronic device is run 3 times. Determine the probability that

(i) the number 6 will be generated exactly twice. (2 marks)

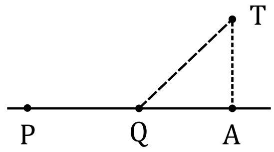
Solution
$Y \sim B(3, 0.2)$ $P(Y=2)=0.096$
Specific behaviours
✓ indicates correct method ✓ probability

(ii) the sum of the numbers generated is at least 23. (3 marks)

Solution
Require $10, 10, 10$ or $10, 10, 6$ or $10, 10, 3$ in any order. $$P=0.3^3+3(0.3)^2(0.2)+3(0.3)^2(0.45)$$ $$=\frac{81}{400}=0.2025$$
Specific behaviours
✓ indicates required events ✓ indicates correct probabilities for at least two events ✓ correct probability

Question 19 **(8 marks)**

An offshore wind turbine T lies 9 km away from the nearest point A on a straight coast. It must be connected to a power storage facility P that lies on the coast 40 km away from A.



Engineers will lay the cable in two straight sections, from T to Q, where Q is a point on the coast x km from A, and then from Q to P.

The cost of installing cable along the coastline is $\$4000$ per km and offshore is $\$5000$ per km.

(a) Determine, to the nearest hundred dollars, the cost of installing the cable when Q lies midway from A to P. (2 marks)

Solution

$$C = 4000 \times 20 + 5000 \times \sqrt{20^2 + 9^2}$$
$$= \$189\,700$$

Specific behaviours

✓ correct expression
✓ calculates cost

(b) Show that C, the cost in thousands of dollars, to run the cable from T to Q to P, is given by $C = 5\sqrt{x^2 + 81} - 4x + 160$. (2 marks)

Solution

$$C_{TQ} = 5 \times QT = 5 \times \sqrt{x^2 + 9^2}$$
$$C_{QP} = 4(40 - x) = 160 - 4x$$

Hence

$$C = C_{TQ} + C_{QP} = 5\sqrt{x^2 + 81} - 4x + 160$$

Specific behaviours

✓ expression for cable from T to Q
✓ expression for cable from Q to P and shows sum

(c) Use calculus techniques to determine, with justification, the minimum cost of laying the cable from T to P. (4 marks)

Solution

$$C'(x) = \frac{5x}{\sqrt{x^2 + 81}} - 4$$

$$C'(x) = 0 \Rightarrow x = 12$$

$$C(12) = 187$$

$$C''(12) \approx 0.12 \Rightarrow \text{minimum, as +ve concavity}$$

Hence minimum cost is $\$187\,000$.

Specific behaviours

✓ correct derivative
✓ solves for optimum value of x
✓ justifies minimum
✓ states minimum cost

Question 20 **(8 marks)**

Small body P moves in a straight line with acceleration a cm/s^2 at time t s given by

$$a = At + B$$

Initially, P has a displacement of 8 cm relative to a fixed point O and is moving with a velocity of 4 cm/s. Three seconds later, P has a displacement of 3.8 cm and a velocity of -5.9 cm/s.

(a) Determine the value of the constant A and the value of the constant B. (6 marks)

Solution
Velocity: $$v = \int At + B\,dt$$ $$v(t) = \frac{At^2}{2} + Bt + c$$ $$v(0) = 4 \Rightarrow c = 4$$ Displacement: $$s(t) = \int \frac{At^2}{2} + Bt + 4\,dt$$ $$s(t) = \frac{At^3}{6} + \frac{Bt^2}{2} + 4t + k$$ $$s(0) = 8 \Rightarrow k = 8$$ $$v(3) = 4.5A + 3B + 4 = -5.9$$ $$s(3) = 4.5A + 4.5B + 20 = 3.8$$ Solve: $$A = 0.6, \qquad B = -4.2$$
Specific behaviours
✓ antiderivative for velocity, constant evaluated ✓ integral for displacement ✓ displacement, constant evaluated ✓ expressions for $v(3)$ and $s(3)$ ✓ value of A ✓ value of B

(b) Determine the minimum velocity of P. (2 marks)

Solution
$$a = 0 \Rightarrow 0.6t - 4.2 = 0 \Rightarrow t = 7$$ $$v(7) = -10.7\text{ cm/s}$$
Specific behaviours
✓ indicates time for minimum ✓ correct minimum velocity

Question 21 **(5 marks)**

(a) Determine the value of the constant a and the value of the constant b that make each of the following statements true, given that $f(x)$ is a polynomial:

(i) $\int_2^3 f(x)\,dx + \int_{-1}^2 f(x)\,dx = \int_a^b f(x)\,dx.$ (1 mark)

Solution
$a = -1, \qquad b = 3$
Specific behaviours
✓ correct values

(ii) $\int_{-3}^2 f(x)\,dx + \int_2^a f(x)\,dx - \int_{-3}^b f(x)\,dx = \int_2^4 f(x)\,dx.$ (2 marks)

Solution
$a = 4, \qquad b = 2$
Specific behaviours
✓ value of a
✓ value of b

(b) Given $F(x) = \int_a^x (2t-3)dt$, determine the value of x that minimises $F(x)$ and justify it is a minimum. (2 marks

Solution
$F'(x) = \frac{d}{dx}\int_0^x (2t-3)dt = 2x-3$ $2x-3=0 \Rightarrow x = \frac{3}{2}$ $F''(x) = 2 \Rightarrow \text{min}$
Specific behaviours
✓ determines value of x
✓ justifies minimum

End of questions